Available online at www.sciencedirect.com
ScienceDirect
Journal of Approximation Theory 161 (2009) 617–633
Journal of Approximation Theory
www.elsevier.com/locate/jat

Density results with linear combinations of translates of fundamental solutions

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Received 16 March 2008; received in revised form 26 September 2008; accepted 24 November 2008
Available online 3 December 2008

Communicated by Robert Schaback

Abstract

In the present work, we investigate the approximability of solutions of elliptic partial differential equations in a bounded domain Ω by linear combinations of translates of fundamental solutions of the underlying partial differential operator. The singularities of the fundamental solutions lie outside of $\overline{\Omega}$. The domains under consideration may possess holes and they are required to satisfy a rather mild boundary regularity requirement, namely the segment condition. We study approximations with respect to the norms of the spaces $C^k(\overline{\Omega})$ and the spaces of uniformly Hölder continuous functions $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$, and we establish density and non-density results for elliptic operators with constant coefficients. We also provide applications of our density results related to the method of fundamental solutions and to the theory of universal series.

Keywords: Fundamental solutions; Elliptic partial differential equations; Elliptic systems; Approximation by special functions; Method of fundamental solutions; Universal series

1. Introduction

Let $\mathcal{L} = \sum_{|\alpha|\le m} a_\alpha D^\alpha$ be an elliptic operator with C^∞-coefficients possessing a fundamental solution. Browder [6] showed in 1962 that linear combinations of fundamental solutions of $\mathcal{L}$ with singularities in an arbitrary open set outside the closure of a bounded open domain Ω without holes are dense, in the sense of the uniform norm, in the space

$$\mathcal{X} = \left\{u \in C^m(\Omega) : \mathcal{L}u = 0 \text{ in } \Omega\right\} \cap C(\overline{\Omega}).$$

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0021-9045/$ - see front matter
doi:10.1016/j.jat.2008.11.018

The domain Ω is assumed to satisfy the cone condition.[1] Weinstock [32] extended Browder's theorem by showing that the solutions of $\mathcal{L}u = 0$ in Ω, which are also elements of $C^k(\overline{\Omega})$, can be approximated by solutions of $\mathcal{L}u = 0$ in a neighborhood of $\overline{\Omega}$, when $0 \leq k < m$, where m is the order of $\mathcal{L}$. In Weinstock's work, $\mathcal{L}$ is assumed to be an elliptic operator with constant coefficients and the domain Ω is required to satisfy a weaker condition, the segment condition (see Definition 4). Detailed surveys on the extensions of Browder's work and approximations of elliptic equations by particular solutions of the same equation can be found in [27,31] and in references therein. In this work we extend Weinstock's theorem to more general norms, and we study approximations by linear combinations of fundamental solutions with singularities outside of $\overline{\Omega}$. Our density results are with respect to the spaces $C^k(\overline{\Omega})$, $k \in \mathbb{N}$ and the spaces of uniformly Hölder continuous functions $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$, $k \in \mathbb{N}$, $\sigma \in (0, 1)$. In our results the domain Ω may possess holes or equivalently its complement may be disconnected, and the elliptic operators under consideration are with constant coefficients. We observe that analogous density results *do not hold* with respect to the spaces $W^{k,\infty}(\Omega)$ and $\mathrm{Lip}^{k,\sigma}(\overline{\Omega})$. (The spaces $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$ and $\mathrm{Lip}^{k,\sigma}(\overline{\Omega})$ are defined in 2.1.) We also provide density results for the case in which the singularities of the fundamental solutions lie on the boundary of a domain Ω' *embracing* Ω- Such results establish the applicability of the *method of fundamental solutions*. Finally, we present applications of our density results related to the theory of *universal series*. In particular, we establish the existence of a universal series of translates of fundamental solutions with coefficients in $\bigcap_{p>1} l^p(\mathbb{N})$ and the non-existence of such a series in $l^1(\mathbb{N})$.

The paper is organized as follows. In Section 2 we give definitions of our function spaces, we describe the method of fundamental solutions and we provide definitions and the main result of the abstract theory of universal series. Section 3 contains the main density results. In Section 4 we provide applications of our density results in the theory of universal series. We also provide variations of these results which establish the applicability of the method of fundamental solutions for certain elliptic boundary value problems. In order to avoid overloading the main text, certain proofs are relegated to the Appendix.

2. Preliminaries

2.1. Function spaces

The spaces $C^k(\overline{\Omega})$. If Ω is an open bounded domain in $\mathbb{R}^n$, then the space $C^k(\Omega)$, where k is a non-negative integer, contains all functions u which, together with all their partial derivatives $D^\alpha u$ of orders $|\alpha| \leq k$, are continuous in Ω and $C^\infty(\Omega) = \bigcap_{k\in\mathbb{N}} C^k(\Omega)$. The space $C^k(\overline{\Omega})$ consists of all functions $u \in C^k(\Omega)$ for which $D^\alpha u$ is uniformly continuous and bounded in Ω for all $|\alpha| \leq k$, and thus extend continuously to $\overline{\Omega}$. In fact, $C^k(\overline{\Omega})$ is a Banach space with norm $|u|_k = \max_{|\alpha|\leq k} \max_{\boldsymbol{x}\in\overline{\Omega}} |D^\alpha u(\boldsymbol{x})|$.

[1] The following boundary requirement is stronger than the segment condition [2]:

Definition 1 (*The Cone Condition*). Let Ω be an open subset of $\mathbb{R}^n$. We say Ω satisfies the *cone condition* if there are a finite open cover $\{U_j\}_{j=1}^J$ of $\partial\Omega$ and an $h > 0$, such that, for every $x \in \Omega \cup U_j$, there is a unit vector $\boldsymbol{\xi}_j \in \mathbb{R}^n$ such that the cone $C_h(\boldsymbol{\xi}_j) = \Big\{\boldsymbol{y} = \boldsymbol{x} + r\boldsymbol{\xi} : r \in (0, h) \text{ and } |\boldsymbol{\xi} - \boldsymbol{\xi}_j| < h\Big\}$, is a subset of Ω.

Hölder spaces. Let now $\sigma \in (0, 1)$ and Ω be an open bounded domain in $\mathbb{R}^n$. The space of Hölder continuous functions $\mathrm{Lip}^{0,\sigma}(\overline{\Omega})$ consists of all functions u, such that

$$[u]_\sigma = \sup_{\delta>0} \omega_\sigma(u, \delta) < \infty, \quad \text{where } \omega_\sigma(u, \delta) = \sup_{\substack{\boldsymbol{x},\boldsymbol{y}\in\overline{\Omega} \\ 0<|\boldsymbol{x}-\boldsymbol{y}|<\delta}} \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|^\sigma}.$$

The space $\mathrm{Lip}^{0,\sigma}(\overline{\Omega})$ is a Banach space with norm $|u|_{0,\sigma} = |u|_0 + [u]_\sigma$, where $|\cdot|_0$ is the norm of $C(\overline{\Omega})$. In general, if $k \in \mathbb{N}$, then the space $\mathrm{Lip}^{k,\sigma}(\overline{\Omega})$ consists of all functions u which, together with all their partial derivatives $D^\alpha u$, $|\alpha| \le k$, belong to $\mathrm{Lip}^{0,\sigma}(\overline{\Omega})$. The space $\mathrm{Lip}^{k,\sigma}(\overline{\Omega})$ is a Banach space with respect to the norm given by $|u|_{k,\sigma} = |u|_k + \max_{|\alpha|\le k}[D^\alpha u]_\sigma$. If $u \in \mathrm{Lip}^{k,\sigma}(\overline{\Omega})$ and $\lim_{\delta\to 0} \omega_\delta(u, \delta) = 0$, then u is called *uniformly Hölder continuous of order* (k, σ) *in* $\overline{\Omega}$. The set of uniformly Hölder continuous of order (k, σ) in $\overline{\Omega}$, which is denoted by $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$, is a closed subspace of $\mathrm{Lip}^{k,\sigma}(\overline{\Omega})$ and thus a Banach space as well. It is readily seen that, if $u \in \mathrm{Lip}^{k,\sigma}(\overline{\Omega})$ can be approximated in the $|\cdot|_{k,\sigma}$-norm by functions which are C^∞ in a neighborhood of $\overline{\Omega}$, then $u \in \mathrm{lip}^{k,\sigma}(\overline{\Omega})$. Thus, the elements of $\mathrm{Lip}^{k,\sigma}(\overline{\Omega}) \smallsetminus \mathrm{lip}^{k,\sigma}(\overline{\Omega})$ cannot be approximated by functions, which are C^∞ in a neighborhood of $\overline{\Omega}$ [27, Theorem 1.3.6]. We extend the definition of the space $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$, for $\sigma = 0$, by setting $\mathrm{lip}^{k,0}(\overline{\Omega}) = C^k(\overline{\Omega})$. Note that, $\mathrm{Lip}^{k,\sigma}(\overline{\Omega})$ is the standard Hölder space, and it is usually denoted as $C^{k,\sigma}(\overline{\Omega})$.

Test functions and distributions. If Ω is an open domain in $\mathbb{R}^n$, then $\mathscr{D}(\Omega)$ is the set of test functions in Ω, i.e., infinitely differentiable functions in Ω with compact support, and $\mathscr{D}'(\Omega)$ the set of distributions on Ω. Note that, $\mathscr{D}(\Omega)$ is a locally convex topological vector space, when endowed with its usual inductive limit topology, and $\mathscr{D}'(\Omega)$ is its dual. If $u, v \in \mathscr{D}'(\mathbb{R}^n)$, one of which (say v) has compact support, then their convolution defines a distribution by $\langle \psi, u * v\rangle = \langle \check{v} * \psi, u\rangle$, where $\psi \in \mathscr{D}(\mathbb{R}^n)$. The distribution $\check{v}$ is defined by $\check{v}(\varphi) = v(\check{\varphi})$, where $\check{\varphi}(\boldsymbol{x}) = \varphi(-\boldsymbol{x})$. (For further details see [22].)

2.2. Fundamental solutions

Let $\mathcal{L} = \sum_{|\alpha|\le m} a_\alpha D^\alpha$ be a partial differential operator in $\mathbb{R}^n$ with constant coefficients of order m. A fundamental solutions of $\mathcal{L}$ is a function $e : \mathbb{R}^n \smallsetminus \{0\}$ satisfying $\mathcal{L}e = \delta$, where δ is the Dirac measure with unit mass at the origin, in the sense of distributions, i.e.,

$$(\mathcal{L}e)(\psi) = \int_{\mathbb{R}^n} e(\boldsymbol{x})\mathcal{L}^\star\psi(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \psi(0) = \delta(\psi),$$

for every $\psi \in C_0^\infty(\mathbb{R}^n)$, where $\mathcal{L}^\star u = \sum_{|\alpha|\le m}(-1)^{|\alpha|}a_\alpha D^\alpha u$. The operator $\mathcal{L}^\star$ is known as the *adjoint* of $\mathcal{L}$. Also, if $\mathcal{L}$ is elliptic, then e is real analytic in $\mathbb{R}^n \smallsetminus \{0\}$ and satisfies, in the classical sense, the equation $\mathcal{L}e(\boldsymbol{x}) = 0$, for every $\boldsymbol{x} \in \mathbb{R}^n \smallsetminus \{0\}$. It is noteworthy that, if e is a fundamental solution of the partial differential operator $\mathcal{L}$ and $\nu \in \mathscr{D}'(\Omega)$ has compact support, then we have

$$\mathcal{L}(e * \nu) = (\mathcal{L}e) * \nu = \delta * \nu = \nu,$$

in the sense of distributions. Malgrange [19] and Ehrenpreis [8] independently established in 1955–56 the existence of fundamental solutions for partial differential operators with constant coefficients.

2.3. Universal series

Let X be a Banach space on $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) and $\{u_\ell\}_{\ell\in\mathbb{N}} \subset X$.

Definition 2. An element $\boldsymbol{a} = \{a_\ell\}_{\ell\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ belongs to the set $\mathcal{U}$ if the sequence of partial sums $\{\sum_{j=0}^{\ell} a_j u_j\}_{\ell\in\mathbb{N}}$ is dense in X. The set $\mathcal{U}$ is the class of *unrestricted* universal series.

Clearly, if $\boldsymbol{a} = \{a_\ell\}_{\ell\in\mathbb{N}}$ is a universal series, then for every $u \in X$, there exists an increasing sequence $\{\lambda_\ell\}_{\ell\in\mathbb{N}} \subset \mathbb{N}$, such that $\lim_{\ell\to\infty} \sum_{j=0}^{\lambda_\ell} a_j u_j = u$.

2.3.1. Restricted universal series

Of interest is whether universal series exist in specific subspaces of $\mathbb{K}^{\mathbb{N}}$. Let A be a linear subspace of $\mathbb{K}^{\mathbb{N}}$ which is a Fréchet space on $\mathbb{K}$ and satisfies the following postulates:

[P_1] The projections $\pi_\ell : A \to \mathbb{K}$, where $\pi_\ell\left(\{a_j\}_{j\in\mathbb{N}}\right) = a_\ell$, are continuous, for all $\ell \in \mathbb{N}$.
[P_2] Let $G = \{\{a_\ell\}_{\ell\in\mathbb{N}} \in A : a_\ell \neq 0$ holds only for finitely many $\ell \in \mathbb{N}\}$. Then $G \subset A$.
[P_3] Let $\{\boldsymbol{e}^\ell\}_{\ell\in\mathbb{N}} \subset \mathbb{K}^{\mathbb{N}}$, where $\boldsymbol{e}^\ell = (\delta_{j\ell})_{j\in\mathbb{N}}$. Then $\lim_{\ell\to\infty} \sum_{j=0}^{\ell} a_j \mathbf{e}^j = \boldsymbol{a}$, with respect to the distance of A, for every $\boldsymbol{a} = \{a_\ell\}_{\ell\in\mathbb{N}} \in A$.

Then $\mathcal{U}_A = \mathcal{U} \cap A$ is the class of *restricted* universal series.

The following proposition is the main result of the abstract theory of universal series:

Proposition 1. *Let A be a Fréchet space satisfying the postulates* [P_1]–[P_3]. *Let also $d(\cdot,\cdot)$ be the distance in A and $\|\cdot\|_X$ the norm of X. Then the following are equivalent:*

(i) $\mathcal{U}_A \neq \varnothing$.
(ii) *For every $u \in X$ and $\varepsilon > 0$, there exist $\ell \in \mathbb{N}$ and $c_0, \ldots, c_\ell \in K$ such that*

$$\|c_0u_0 + \cdots + c_\ell u_\ell - u\|_X < \varepsilon \quad \textit{and} \quad d(c_0\boldsymbol{e}^0 + \cdots + c_\ell \boldsymbol{e}^\ell, 0) < \varepsilon.$$

(iii) $\mathcal{U}_A$ *is a dense G_δ in A.*
(iv) $\mathcal{U}_A \cup \{\mathbf{0}\}$ *contains a dense linear subspace of A.*

Proof. See [4,20]. □

2.4. The method of fundamental solutions: A Trefftz method

Let $\mathcal{L} = \sum_{|\alpha|\le m} a_\alpha D^\alpha$ be an elliptic operator in $\mathbb{R}^n$. In Trefftz methods, the solution of the boundary value problem

$$\mathcal{L}u = 0 \quad \text{in } \Omega, \tag{2.1a}$$

$$\mathcal{B}u = \boldsymbol{f} \quad \text{on } \partial\Omega, \tag{2.1b}$$

where $\Omega \subset \mathbb{R}^n$ is open and $\mathcal{B}u = \boldsymbol{f}$ is the boundary condition, is approximated by linear combinations of particular solutions of (2.1a), provided that such linear combinations are dense in the set of all solutions of this equation. In the method of fundamental solutions (MFS), which was introduced by Kupradze and Aleksidze [18] in 1963, the particular solutions are the fundamental solutions of $\mathcal{L}$. In a typical application of the MFS for a boundary value problem (Dirichlet, Neumann or of mixed type) for Laplace's equation in a bounded domain Ω, the solution is approximated by a finite linear combination of the form

$$u^N(\boldsymbol{x};\boldsymbol{c}) = \sum_{j=1}^{N} c_j e_1(\boldsymbol{x} - \boldsymbol{y}_j), \tag{2.2}$$

where, $N \in \mathbb{N}$, $\boldsymbol{c} = (c_j)_{j=1}^N \subset \mathbb{R}^N$ and $\{\boldsymbol{y}_j\}_{j=1}^N$, the singularities, are located on a *pseudo-boundary*, i.e., a prescribed boundary $\partial\Omega'$ of a domain Ω' *embracing* Ω (see Definition 3). The function e_1 is given by

$$e_1(\boldsymbol{x}) = \begin{cases} -\dfrac{\log|\boldsymbol{x}|}{2\pi}, & \text{if } n = 2, \\ -\dfrac{|\boldsymbol{x}|^{2-n}}{(2-n)\omega_{n-1}}, & \text{if } n > 2, \end{cases} \tag{2.3}$$

and it is a fundamental solution of the operator $-\Delta$, where ω_{n-1} is the area of the surface of the unit sphere S^{n-1} in $\mathbb{R}^n$ and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$ (see [5,9,23]).

Definition 3 (*The Embracing Pseudo-boundary*). Let Ω, Ω' be open connected subsets of $\mathbb{R}^n$. We say that Ω' *embraces* Ω if $\overline{\Omega} \subset \Omega'$, and for every connected component V of $\mathbb{R}^n \setminus \overline{\Omega}$, there is an open connected component V' of $\mathbb{R}^n \setminus \overline{\Omega}'$ such that $\overline{V}' \subset V$.

In the most popular perhaps formulation of the MFS, the coefficients in (2.2) can be obtained by collocation of the boundary data. In the case of the Dirichlet problem for Laplace's equation this is done by choosing N points $\{\boldsymbol{y}_j\}_{j=1}^N$ on $\partial\Omega'$, the singularities, and N collocation points $\{\boldsymbol{x}_k\}_{k=1}^N$ on $\partial\Omega$, and require that the approximate solution u^N satisfies

$$u^N(\boldsymbol{x}_k; \boldsymbol{c}) = \sum_{j=1}^N c_j e_1(\boldsymbol{x}_k - \boldsymbol{y}_j) = f(\boldsymbol{x}_k), \quad k = 1, \ldots, N. \tag{2.4}$$

The coefficients $\{c_j\}_{j=1}^N$ can be determined uniquely, provided the system matrix $G = \big(e_1(\boldsymbol{x}_k - \boldsymbol{y}_j)\big)_{k,j=1}^N$, is non-singular. If $\Omega = D_\varrho$, the disk of radius ϱ, and with the circumference of a concentric disk as a pseudo-boundary, it is shown that the supremum error in the MFS approximation tends to zero as N tends to infinity, provided that both singularities and collocation points are uniformly distributed on $\partial\Omega$ and $\partial\Omega'$, and the rate of convergence increases as the smoothness of the data improves. In particular, if the boundary data belong to $C^\ell(\partial D_\varrho)$, then the error is $\mathcal{O}(N^{-\ell+1})$. (See [14,26,24].) In [15], this result is generalized to regions in the plane whose boundaries are analytic Jordan curves, while in [29], the same result is obtained for annular regions. Convergence of the MFS for the Helmholtz equation in the exterior of a disk was established in [30].

In the last years, the MFS has become a popular meshless technique for the solution of elliptic boundary value problems because of the ease with which it can be implemented and the fact that it does not require an elaborate discretization of the boundary, and it does not involve potentially troublesome and costly integrations over the boundary. Also, it can be applied even in the case of domains with holes, corners and cusps, and the evaluation of the approximate solution at interior points can be carried out directly. (Unlike the boundary element method for which a quadrature is needed.) Furthermore, the derivatives of the MFS approximation can also be evaluated directly. For further details on the merits and shortcomings of the MFS see [9].

Comprehensive reference lists of applications of the MFS can be found in [3,10,12,13,16]. The applicability of the MFS, i.e., the question whether linear combinations of fundamental solutions with singularities lying on a prescribed pseudo-boundary are dense in the set of all solutions of the corresponding equation has been studied in [5,17,18,23,25].

3. Density results

3.1. The main result

The domains in our density results satisfy a rather weak boundary regularity requirement, namely the segment condition:

Definition 4 (*The Segment Condition*). Let Ω be an open subset of $\mathbb{R}^n$. We say that Ω satisfies the *segment condition* if every $\boldsymbol{x}$ on the boundary $\partial\Omega$ of Ω possesses a neighborhood $U_{\boldsymbol{x}}$ and a non-zero vector $\boldsymbol{\xi}_{\boldsymbol{x}}$ such that, if $\boldsymbol{y} \in U_{\boldsymbol{x}} \cap \overline{\Omega}$, then $\boldsymbol{y} + t\boldsymbol{\xi}_{\boldsymbol{x}} \in \Omega$ for every $t \in (0, 1)$.

Note that the segment condition is weaker than the *cone condition* (see Footnote 1) and allows the boundaries to have corners and cusps. Also, the boundary of the domains which satisfy this condition is an $(n-1)$-dimensional manifold. However, if a domain satisfies the segment condition it cannot lie on both sides of any part of its boundary [2]. Therefore, domains with cracks do not satisfy the segment condition. Furthermore, domains satisfying the segment condition coincide with the interior of their closure while bounded domains satisfying the segment condition can have only finitely many holes (i.e., their complement can have finitely many connected components). It is not hard to prove that, if a domain satisfies the segment condition, then every connected component of its complement has a non-empty interior.

In our first density result, the approximation of the solutions of the elliptic equation $\mathcal{L}u = 0$ is achieved by linear combinations of fundamental solutions of $\mathcal{L}$ with singularities lying in an open set outside of $\overline{\Omega}$:

Theorem 1. *Let $\mathcal{L} = \sum_{|\alpha|\le m} a_\alpha D^\alpha$ be an elliptic operator with constant coefficients in $\mathbb{R}^n$ possessing a fundamental solution $e = e(\boldsymbol{x})$. Let also Ω be an open bounded domain satisfying the segment condition and $U \subset \mathbb{R}^n \smallsetminus \overline{\Omega}$ an open set intersecting all the components of $\mathbb{R}^n \smallsetminus \overline{\Omega}$. If k a non-negative integer and $\sigma \in [0, 1)$, then the set $\mathcal{X}$ of linear combinations of the functions $\varphi_{\boldsymbol{y}}(\boldsymbol{x}) = e(\boldsymbol{x} - \boldsymbol{y})$, with $\boldsymbol{y} \in U$, is dense in*

$$\mathcal{Y}_{k,\sigma} = \big\{u \in C^m(\Omega) : \mathcal{L}u = 0\big\} \cap \mathrm{lip}^{k,\sigma}(\overline{\Omega}),$$

with respect to the norm of $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$.

Proof. We follow the duality argument of the proof of Theorem 3 in [6].

Both $\mathcal{X}$ and $\mathcal{Y}_{k,\sigma}$ are linear subspaces of $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$. From the Hahn–Banach Theorem, it suffices to show that $\mathcal{X}^\perp \subset \mathcal{Y}_{k,\sigma}^\perp$, i.e.,

$$\text{if } \nu \in \Big(\mathrm{lip}^{k,\sigma}(\overline{\Omega})\Big)' \quad \text{and} \quad \underset{\text{for every } u\in\mathcal{X}}{\langle u, \nu\rangle = 0} \quad \text{then} \quad \underset{\text{for every } u\in\mathcal{Y}_{k,\sigma}}{\langle u, \nu\rangle = 0.}$$

Let $\nu \in \big(C^k(\overline{\Omega})\big)'$ annihilating $\mathcal{X}$, i.e., $\langle u, \nu\rangle = 0$, for every $u \in \mathcal{X}$. Clearly, if $\boldsymbol{x} \in U$, then the function $u(\boldsymbol{y}) = e(\boldsymbol{y} - \boldsymbol{x}) = (\tau_{\boldsymbol{x}} e)(\boldsymbol{y})$ belongs to $\mathcal{X}$ and

$$0 = \langle u, \nu\rangle = \langle \tau_{\boldsymbol{x}} e, \nu\rangle = (\check{e} * \nu)(\boldsymbol{x}),$$

where $\check{e}(\boldsymbol{x}) = e(-\boldsymbol{x})$. The functional ν defines also a distribution in $\mathbb{R}^n$. Also, $\vartheta = \check{e} * \nu$ vanishes in U. Note that ϑ defines a distribution in $\mathbb{R}^n$, as a convolution of two distributions, one of which (namely ν) has compact support ($\operatorname{supp}\nu \subset \overline{\Omega}$). Since $\check{e}$ is a fundamental solution of $\mathcal{L}^\star$ (the adjoint of $\mathcal{L}$), then $\mathcal{L}^\star\vartheta = \nu$, in the sense of distributions. Meanwhile, ϑ satisfies the elliptic equation $\mathcal{L}^\star u = 0$ in $\mathbb{R}^n \smallsetminus \overline{\Omega}$, and thus it is a real analytic function in $\mathbb{R}^n \smallsetminus \overline{\Omega}$. Let V be a connected component of $\mathbb{R}^n \setminus \overline{\Omega}$. Since U intersects V, then ϑ vanishes in V, and consequently in the whole $\mathbb{R}^n \setminus \overline{\Omega}$, and thus $\operatorname{supp} \vartheta \subset \overline{\Omega}$.

We now need the following proposition:

Proposition 2. *Let $\mathcal{L}$ be an elliptic operator with constant coefficients in $\mathbb{R}^n$ and $e = e(\boldsymbol{x})$ be a fundamental solution of $\mathcal{L}$. Also, let Ω be an open bounded subset of $\mathbb{R}^n$ satisfying the segment condition and ν is a continuous linear functional on $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$. If $\vartheta = e * \nu$ is the convolution of the distributions e and ν and $\operatorname{supp} \vartheta \subset \overline{\Omega}$, then there exists a sequence $\{\vartheta^\ell\}_{\ell\in\mathbb{N}} \subset \mathscr{D}'(\mathbb{R}^n)$ with $\operatorname{supp} \vartheta^\ell \subset \Omega$ and $\{\mathcal{L}\vartheta^\ell\}_{\ell\in\mathbb{N}} \subset \left(\mathrm{lip}^{k,\sigma}(\overline{\Omega})\right)'$, such that $\{\mathcal{L}\vartheta^\ell\}_{\ell\in\mathbb{N}}$ converges to ν in the weak* sense of $\left(\mathrm{lip}^{k,\sigma}(\overline{\Omega})\right)'$, i.e.,*

$$\lim_{\ell\to\infty} \langle u, \mathcal{L}\vartheta^\ell\rangle = \langle u, \nu\rangle,$$

for every $u \in \mathrm{lip}^{k,\sigma}(\overline{\Omega})$.

Proof. See Appendix. □

Let $u \in \mathcal{Y}_{k,\sigma}$. Then by virtue of Proposition 2, applied to the operator $\mathcal{L}^\star$ which is also elliptic, there exists a sequence of distributions $\{\vartheta^\ell\}_{\ell\in\mathbb{N}}$ supported in Ω with $\{\mathcal{L}^\star\vartheta^\ell\}_{\ell\in\mathbb{N}} \subset \left(\mathrm{lip}^{k,\sigma}(\overline{\Omega})\right)'$, such that

$$\langle u, \nu\rangle = \lim_{\ell\to\infty} \langle u, \mathcal{L}^\star\vartheta^\ell\rangle.$$

It suffices to show that $\langle u, \mathcal{L}^\star\vartheta^\ell\rangle = 0$, for every $\ell \in \mathbb{N}$. Let $\zeta \in \mathscr{D}(\Omega)$ which is equal to 1 in a neighborhood of the support of ϑ^ℓ. Then functions u and ζu agree in a neighborhood of the $\operatorname{supp} \vartheta^\ell$, and thus $\langle u, \mathcal{L}^\star\vartheta^\ell\rangle = \langle \zeta u, \mathcal{L}^\star\vartheta^\ell\rangle$. On the other hand, $\zeta u \in \mathscr{D}(\Omega)$, since u is real analytic in Ω, then according to the definition of the distribution $\mathcal{L}^\star\vartheta^\ell$:

$$\langle \zeta u, \mathcal{L}^\star\vartheta^\ell\rangle = \langle \mathcal{L}(\zeta u), \vartheta^\ell\rangle.$$

The right-hand side in the above equality is equal to zero since $\mathcal{L}(\zeta u) = \mathcal{L}u = 0$ in a neighborhood of the support of ϑ^ℓ. Therefore

$$\langle u, \mathcal{L}^\star\vartheta^\ell\rangle = \langle \mathcal{L}(\zeta u), \vartheta^\ell\rangle = 0,$$

and thus

$$\langle u, \nu\rangle = \lim_{\ell\to\infty} \langle u, \mathcal{L}^\star\vartheta^\ell\rangle = 0,$$

which concludes the proof of Theorem 1. □

Remark 3.1. Analogous density results are obtainable with respect to Sobolev norms, provided that Ω satisfies the *cone condition* (see Footnote 1). On the other hand, such density results do not hold for the space $\mathrm{Lip}^{k,\sigma}(\overline{\Omega})$. This is because of the fact that the set $\mathcal{X}$, as defined in Theorem 1, is a subset of $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$, and since $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$ is a closed subspace of $\mathrm{Lip}^{k,\sigma}(\overline{\Omega})$, then the closure of $\mathcal{X}$, with respect to the norm $|\cdot|_{k,\sigma}$, is a subset of $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$.

3.2. Uniform approximation by solutions of elliptic equations

Let Ω be an open bounded subset of $\mathbb{R}^n$ and $\mathcal{L}$ be an elliptic operator with constant coefficients of order m. Let also, k be a non-negative integer and $\sigma \in [0, 1)$. We define as $\mathcal{H}^{k,\sigma}_{\mathcal{L}}(\overline{\Omega})$ the space of functions in $\overline{\Omega}$ which can be approximated by functions which are solutions of the equation $\mathcal{L}u = 0$ in a neighborhood of $\overline{\Omega}$, with respect to the $\mathrm{lip}^{k,\sigma}$-norm in $\overline{\Omega}$. The result that follows, which is an extension of Proposition 5 in [32], shows that membership in $\mathcal{H}^{k,\sigma}_{\mathcal{L}}(\overline{\Omega})$ is a local property.

Theorem 2. *Let $k \in \mathbb{N}$, $\sigma \in [0, 1)$ and $\Omega \subset \mathbb{R}^n$ be open and bounded. If $u \in \mathrm{lip}^{k,\sigma}(\overline{\Omega})$ and if for every $\boldsymbol{x} \in \overline{\Omega}$, there exists an open neighborhood $U_{\boldsymbol{x}}$ of $\boldsymbol{x}$ in $\mathbb{R}^n$ such that $u \in \mathcal{H}^{k,\sigma}_{\mathcal{L}}(\overline{\Omega} \cap \overline{U}_{\boldsymbol{x}})$, then $u \in \mathcal{H}^{k,\sigma}_{\mathcal{L}}(\overline{\Omega})$.*

Proof. It is a consequence of Corollary A.1 in the Appendix. □

Let also $\mathcal{B}^{k,\sigma}_{\mathcal{L}}(\overline{\Omega})$ be the space of functions in $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$ which satisfy the equation $\mathcal{L}u = 0$ in Ω. The following is an extension of Propostion 7 in [32].

Theorem 3. *If Ω is an open bounded subset of $\mathbb{R}^n$ satisfying the segment condition and $\mathcal{L}$ is an elliptic operator with constant coefficients of order m, then $\mathcal{B}^{k,\sigma}_{\mathcal{L}}(\overline{\Omega}) = \mathcal{H}^{k,\sigma}_{\mathcal{L}}(\overline{\Omega})$ for every non-negative integer k and every $\sigma \in [0, 1)$.*

Proof. Clearly,

$$\mathcal{H}^{k,\sigma}_{\mathcal{L}}(\overline{\Omega}) \subset \mathcal{B}^{k,\sigma}_{\mathcal{L}}(\overline{\Omega}) = \left\{u \in C^m(\Omega) : \mathcal{L}u = 0\right\} \cap \mathrm{lip}^{k,\sigma}(\overline{\Omega}).$$

Let e be a fundamental solution of $\mathcal{L}$ and V be an open bounded subset of $\mathbb{R}^n \smallsetminus \overline{\Omega}$ intersecting all the components of $\mathbb{R}^n \smallsetminus \overline{\Omega}$. If $\mathcal{X}$ the set of linear combination of the form $\sum_{j=1}^{N} c_j e(\boldsymbol{x} - \boldsymbol{y}_j)$, with $\boldsymbol{y}_j \in V$, then clearly $\mathcal{X}$ is a subset of the set of functions satisfying the equation $\mathcal{L}u = 0$ in a neighborhood of $\overline{\Omega}$. Thus, the closure of $\mathcal{X}$ with respect to the norm of $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$ is a subset of $\mathcal{H}^{k,\sigma}_{\mathcal{L}}(\overline{\Omega})$. Due to Theorem 1 the closure of $\mathcal{X}$ coincides with $\mathcal{B}^{k,\sigma}_{\mathcal{L}}(\overline{\Omega})$, and thus $\mathcal{B}^{k,\sigma}_{\mathcal{L}}(\overline{\Omega}) = \mathcal{H}^{k,\sigma}_{\mathcal{L}}(\overline{\Omega})$. □

4. Applications

4.1. Universal series of fundamental solutions

Let Ω and U be an open bounded domains with Ω satisfying the segment condition and U intersecting all the connected components of $\mathbb{R}^n \smallsetminus \overline{\Omega}$. Let $S = \{\boldsymbol{y}_j\}_{j\in\mathbb{N}} \subset U$ be countable and dense in U. The corresponding function space shall be

$$\mathcal{X} = \left\{u \in C^m(\Omega) : \mathcal{L}u = 0 \text{ in } \Omega\right\} \cap \mathrm{lip}^{k,\sigma}(\overline{\Omega}).$$

Clearly, $\mathcal{X}$ is a Banach space with respect to the norm of $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$, being a closed subset of $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$.

Let e be a fundamental solution of $\mathcal{L}$ and $\chi_j(\boldsymbol{x}) = e(\boldsymbol{x} - \boldsymbol{y}_j) = (\tau_{\boldsymbol{y}_j} e)(\boldsymbol{x})$, where $\boldsymbol{y}_j \in S$.

Question. *Is there an element $\boldsymbol{a} = \{a_\ell\}_{\ell\in\mathbb{N}}$ of the set $\mathbb{K}^{\mathbb{N}}$, such that the sequence of partial sums $\left\{\sum_{j=0}^{\ell} a_j \chi_j\right\}_{\ell\in\mathbb{N}}$ is dense in $\mathcal{X}$?*

The set of such elements $\boldsymbol{a}$ is denoted by $\mathcal{U}$. In particular, we are interested in finding elements $\boldsymbol{a} = \{a_\ell\}_{\ell\in\mathbb{N}}$ in $\mathcal{U}$ belonging to specific subspaces of $\mathbb{K}^{\mathbb{N}}$. We have the following result:

Theorem 4. *Let $\{\chi_j\}_{j\in\mathbb{N}} \subset \mathcal{X}$ be as defined above and $\mathcal{U} \subset \mathbb{K}^{\mathbb{N}}$ be the class of unrestricted universal series corresponding to $\{\chi_j\}_{j\in\mathbb{N}}$. Then $\mathcal{U} \neq \varnothing$. Furthermore,*

(i) $\mathcal{U}_{l^p} = \mathcal{U} \cap l^p(\mathbb{N}) \neq \varnothing$ *for every* $p \in (1,\infty)$. *Moreover,* $\mathcal{U}_{\cap_{p>1} l^p} = \mathcal{U} \cap \left(\cap_{p>1} l^p(\mathbb{N})\right) \neq \varnothing$.
(ii) $\mathcal{U}_{l^1} = \mathcal{U} \cap l^1(\mathbb{N}) = \varnothing$.

Also, the sets $\mathcal{U}$, $\mathcal{U}_{l^p(\mathbb{N})}$ with $p \in (1,\infty)$ and $\mathcal{U}_{\cap_{p>1} l^p(\mathbb{N})}$ are dense G^δ in $\mathbb{K}^{\mathbb{N}}$, $l^p(\mathbb{N})$ and $\cap_{p>1} l^p(\mathbb{N})$, respectively, and they contain a dense vector subspace except zero.

Proof. We use the argument of the proof of Theorem 3 in [21].

Clearly, $\cap_{p>1} l^p(\mathbb{N}) \subset l^q(\mathbb{N}) \subset \mathbb{K}^{\mathbb{N}}$, for every $q \in (1,\infty)$, and since $l^q(\mathbb{N})$, $\cap_{p>1} l^p(\mathbb{N})$, and $\mathbb{K}^{\mathbb{N}}$ satisfy the postulates [P$_1$]–[P$_3$], it suffices to show that $\mathcal{U} \cap \left(\cap_{p>1} l^p(\mathbb{N})\right) \neq \varnothing$. The space $\bigcap_{p>1} l^p(\mathbb{N})$ is a Fréchet space with distance

$$d(\boldsymbol{a},\boldsymbol{b}) = \sum_{j=1}^{\infty} \frac{1}{2^j} \cdot \frac{\|\boldsymbol{a}-\boldsymbol{b}\|_{1+2^{-j}}}{1+\|\boldsymbol{a}-\boldsymbol{b}\|_{1+2^{-j}}},$$

where $\|\cdot\|_p$ is the norm of $l^p(\mathbb{N})$. Let $\varepsilon > 0$ and $u \in \mathcal{X}$. Let $N \in \mathbb{N}$ be such that $\sum_{j=N+1}^{\infty} 2^{-j} < \varepsilon/2$ and M be a sufficiently large positive integer to be defined later. Due to Theorem 1, there exist $\ell \in \mathbb{N}$ and $c_0,\ldots,c_\ell \in \mathbb{K}$ such that $|\sum_{j=0}^{\ell} c_j\chi_j - u|_{k,\sigma} < \varepsilon$, where $|\cdot|_{k,\sigma}$ is the norm of the space $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$. Since S is dense in U, for every $j = 0,\ldots,\ell$ we can find distinct $\chi_{j_1},\ldots,\chi_{j_M}$, close to χ_j, such that

$$\left|\frac{1}{M}\sum_{j=0}^{\ell}\sum_{i=1}^{M} c_j\chi_{j_i} - u\right|_{k,\sigma} < \varepsilon. \tag{4.1}$$

Clearly,

$$\sum_{j=0}^{\ell}\sum_{i=1}^{M} M^{-p}|c_j|^p = M^{1-p}\sum_{j=0}^{\ell}|c_j|^p \to 0,$$

as $M \to \infty$, where $p = 1 + 2^{-N}$. Thus, we can choose sufficiently large M so that $\|\boldsymbol{c}\|_{1+2^{-j}} < \frac{\varepsilon}{2N}$, for $j = 1,\ldots,N$, where $\boldsymbol{c}$ is the finite sequence of c_j/M corresponding to the coefficients χ_{j_i}'s. It follows that $d(\boldsymbol{c},0) < \varepsilon$ and in combination with (4.1) it implies that $\mathcal{U}_{\cap_{p>1} l^p(\mathbb{N})} \neq \varnothing$, due to Proposition 1.

Next we show that $\mathcal{U}_{l^1(\mathbb{N})} = \varnothing$. Let $u \in \mathcal{X}$ and $W \neq \varnothing$ open subset of U, such that $\overline{W} \subset \Omega$. Clearly, $d = \mathrm{dist}\left(\overline{W},\Omega\right) > 0$. If there exist $c_0,\ldots,c_\ell$ such that

$$|c_0| + \cdots + |c_\ell| < \varepsilon \quad \text{and} \quad \left|\sum_{j=0}^{\ell} c_j\chi_j - u\right|_{k,\sigma} < \varepsilon,$$

and let $s = \max_{(\boldsymbol{x},\boldsymbol{y})\in\overline{W}\times\overline{U}}|e(\boldsymbol{x}-\boldsymbol{y})|$, then for $\mathbf{x} \in \overline{W}$ we shall have

$$|u(\boldsymbol{x})| - \varepsilon \leq \left|\sum_{j=0}^{\ell} c_j\chi_j(\boldsymbol{x})\right| \leq \sum_{j=0}^{\ell}|c_j|\cdot|e(\boldsymbol{x}-\boldsymbol{y}_j)| = s\varepsilon.$$

Thus $|u(\boldsymbol{x})| \leq (1+s)\varepsilon$ for every $\mathbf{x}$ in $\overline{W}$ which leads to contradiction since it requires from u to be arbitrarily small in any open W with $\overline{W} \subset \Omega$. $\square$

Remark 4.1. The existence of universal series of translates of fundamental solutions of Laplace's equation with singularities on a prescribed pseudo-boundary was established in [21].

Remark 4.2. Baire's Theorem yields the existence of a series which is universal simultaneously for all spaces $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$, $k \in \mathbb{N}$, $\sigma \in [0,1)$. This is achieved by proving the existence of series which is universal simultaneously for all spaces $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$, $k \in \mathbb{N}$, $\sigma \in [0,1) \cap \mathbb{Q}$.

4.2. Applicability of the method of fundamental solutions

In the MFS, the solution of an elliptic boundary value problem in a bounded domain Ω is approximated by linear combinations of translates of a fundamental solution of the underlying operator $\mathcal{L}$ with singularities lying on a prescribed pseudo-boundary $\partial\Omega'$. However, before applying the MFS, one needs to know that such linear combinations are dense in the solution space of the equation $\mathcal{L}u = 0$ in Ω.

4.2.1. Laplace equation

Unfortunately, such linear combinations are not always dense in the solution space. If, for example, the pseudo-boundary is the unit circle ∂D, then the translates of

$$e_1(\boldsymbol{x}) = -\frac{1}{2\pi} \log |\boldsymbol{x}|,$$

which is a fundamental solution of $\mathcal{L} = -\Delta$, with singularities on $\partial\Omega$, vanish at the origin, and so do their linear combinations. Thus, such linear combinations *are not dense* in the set of harmonic functions in a disk of radius $\varrho < 1$ centered at the origin. Nevertheless, this non-density result can be removed once $e_1(\boldsymbol{x})$ is replaced by $e_1(\boldsymbol{x}/R)$, for R larger than the diameter of Ω. Note that $e_1(\boldsymbol{x}/R)$ is also a fundamental solution of the Laplacian [23]. On the other hand, in dimensions $n \geq 3$, linear combinations of translates of the fundamental solution e_1 of $-\Delta$ given by (2.3), with singularities lying on a prescribed pseudo-boundary are dense in the solution space.

We have the following result:

Theorem 5. *Let Ω, Ω' be open bounded domains in $\mathbb{R}^n$, $n \geq 3$, with Ω satisfying the segment condition and Ω' embracing Ω and let k be a non-negative integer. Then the space $\mathcal{X}$ of finite linear combinations of the form $u^N(\boldsymbol{x};\boldsymbol{c}) = \sum_{j=1}^N c_j e_1(\boldsymbol{x} - \boldsymbol{y}_j)$, where $N \in \mathbb{N}$, $\{c_j\}_{j=1}^N \subset \mathbb{R}$, $\{\boldsymbol{y}_j\}_{j=1}^N \subset \partial\Omega'$ and e_1 is given by (2.3), is dense in*

$$\mathcal{Y}_k = \left\{ u \in C^2(\Omega) : \Delta u = 0 \text{ in } \Omega \right\} \cap \mathrm{lip}^{k,\sigma}(\overline{\Omega}),$$

with respect to the norm of the space $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$.

Sketch of Proof. We describe a proof along the lines of the proof of Theorem 1.

Let $\nu \in \mathrm{lip}^{k,\sigma}(\overline{\Omega})$, such that $\langle u, \nu \rangle = 0$, for every $u \in \mathcal{X}$, then the convolution $\vartheta = e_1 * \nu$ is harmonic in $\mathbb{R}^n \setminus \overline{\Omega}$, since $\check{e}_1 = e_1$, and vanishes on $\partial\Omega'$. Also, $D^\alpha e_1$ vanishes at infinity for every multi-index α and consequently, so does ϑ. If V is a bounded component of $\mathbb{R}^n \setminus \overline{\Omega}$, then there is an open set V', such that $\overline{V}' \subset V$ and $\partial V' \subset \partial\Omega'$. Since ϑ vanishes on $\partial V' \subset \partial\Omega'$, it vanishes in V' as well, due to the maximum principle, and in the whole of V, since ϑ is a real analytic function in V. In the case of the unbounded component of $\mathbb{R}^n \setminus \overline{\Omega}$, using the fact that ϑ vanishes at infinity and on a boundary of an unbounded component we obtain similarly that ϑ vanishes in the whole component, and therefore in $\mathbb{R}^n \setminus \overline{\Omega}$. The rest of the proof in identical to the proof of Theorem 1. □

4.2.2. Poly-Helmholtz equation

If $\mathcal{L}$ is a higher order elliptic operator, i.e., of order $2m \geq 4$, and e is a fundamental solution of $\mathcal{L}$, then linear combinations of the translates of e, with singularities lying on a pseudo-boundary, are not, in general, dense in the solution space of $\mathcal{L}$. (See [23].) In such cases, the MFS approximation contains m different fundamental solutions corresponding to suitable factors of $\mathcal{L}$. For example, in the case of the poly-Helmholtz operator[2]

$$\mathcal{L} = (\Delta - \kappa_1^2) \cdots (\Delta - \kappa_m^2),$$

with $\kappa_i > 0$ and $\kappa_i \neq \kappa_j$, when $i \neq j$, the MFS approximation is of the form

$$u^N(\boldsymbol{x}; \boldsymbol{c}) = \sum_{j=1}^{m} \sum_{\ell=1}^{N} c_\ell^j e_1(\boldsymbol{x} - \boldsymbol{y}_\ell, \kappa_j^2), \tag{4.2}$$

where $e_1(\cdot, \kappa^2)$ is a fundamental solution of $\Delta - \kappa^2$ given by

$$e_1\left(\boldsymbol{x}, \kappa^2\right) = \begin{cases} -\dfrac{K_0(\kappa|\boldsymbol{x}|)}{2\pi} & \text{if } n = 2, \\ -\dfrac{\mathrm{e}^{-\kappa|\boldsymbol{x}|}}{4\pi|\boldsymbol{x}|} & \text{if } n = 3, \end{cases} \tag{4.3}$$

where $K_0(r)$ is the modified Bessel function of the second kind.

The applicability of the MFS is established by the density result that follows.

Theorem 6. *Let Ω, Ω' be open bounded domains in $\mathbb{R}^n$, $n = 2, 3$, with Ω satisfying the segment condition and Ω' embracing Ω, and let k be a non-negative integer and $\sigma \in [0, 1)$. Further, assume that $\kappa_i > 0$ and $\kappa_i \neq \kappa_j$, when $i \neq j$. Then the space $\mathcal{X}$ of finite linear combinations of the form (4.2) where $e_1(\cdot, \kappa^2)$ is given by (4.3) and $\{\boldsymbol{y}_\ell\}_{\ell=1}^N \subset \partial\Omega'$, is dense in*

$$\mathcal{Y}_{k,\sigma} = \left\{u \in C^{2m}(\Omega) : \mathcal{L}u = 0 \text{ in } \Omega\right\} \cap \mathrm{lip}^{k,\sigma}(\overline{\Omega}),$$

with respect to the norm of $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$, where $\mathcal{L} = (\Delta - \kappa_1^2) \cdots (\Delta - \kappa_m^2)$.

Proof. We assume that $m = 2$. The case $m > 2$ can be done inductively. Let $\kappa, \lambda \in \mathbb{R}^+$ with $k \neq \lambda$. A fundamental solution of $\mathcal{L} = (\Delta - \kappa^2)(\Delta - \lambda^2)$ is

$$e_2(\boldsymbol{x}) = \frac{e_1(\boldsymbol{x}, \lambda^2) - e_1(\boldsymbol{x}, \kappa^2)}{\lambda^2 - \kappa^2}.$$

Similarly, one can construct a fundamental solution of $\mathcal{L} = (\Delta - \kappa_1^2) \cdots (\Delta - \kappa_m^2)$ as a linear combination of $e_1(\cdot, \kappa_1^2), \ldots, e_1(\cdot, \kappa_m^2)$. (See [7,23,28].) It is readily shown that

$$(\Delta - \lambda^2)e_2 = e_1(\cdot, \kappa^2),$$

in the sense of distributions. As in the proof of Theorem 1, let $\nu \in \left(\mathrm{lip}^{k,\sigma}(\overline{\Omega})\right)'$ annihilating $\mathcal{X}$. Then $\nu\left(\tau_{\boldsymbol{x}} e_1(\cdot, \kappa^2)\right) = 0$ and $\nu(\tau_{\boldsymbol{x}} e_2) = 0$, for every $\boldsymbol{x} \in \partial\Omega'$. Thus the convolutions $\vartheta_1 = e_1(\cdot, \kappa^2) * \nu$ and $\vartheta_2 = e_2 * \nu$ vanish on $\partial\Omega'$, and $(\Delta - \lambda^2)\vartheta_2 = \vartheta_1$, in the sense of

[2] The poly-Helmholtz operator, which is elliptic, arises from m-porosity media as well as from m-layered aquifer systems. See [7] and references therein.

distributions in $\mathbb{R}^n$. The distribution ϑ_1 is a real analytic function and satisfies the maximum principle in $\mathbb{R}^n \setminus \overline{\Omega}$, since it satisfies the equation $(\Delta - \kappa^2)u = 0$. (See [11].) Also, $e_1(\cdot, \kappa^2)$ together with its partial derivatives of all orders vanish at infinity. (See [1, p. 374–378].) As in the proof of Theorem 5, these imply that ϑ_1 vanishes in $\mathbb{R}^n \setminus \overline{\Omega}$. Consequently, $(\Delta - \lambda^2)\vartheta_2 = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$. Repeating the previous argument we obtain that ϑ_2 vanishes in $\mathbb{R}^n \setminus \overline{\Omega}$. The rest of the proof is identical to the proof of Theorem 1. □

Remark 4.3. It is noteworthy that Theorem 5 possesses a continuous analogue in which the linear combinations of translates of fundamental solutions of the form $u^N(\boldsymbol{x}; \boldsymbol{c}) = \sum_{j=1}^N c_j e_1(\boldsymbol{x} - \boldsymbol{y}_j)$ are replaced by expressions of the form [5]

$$u(\boldsymbol{x};\, f) = \int_{\partial\Omega'} e_1(\boldsymbol{x} - \boldsymbol{y}) f(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}, \qquad f \in C(\partial\Omega').$$

In the case of Theorem 6, linear combinations of the form

$$u^N(\boldsymbol{x}; \boldsymbol{c}) = \sum_{j=1}^{m} \sum_{\ell=1}^{N} c_\ell^j e_1(\boldsymbol{x} - \boldsymbol{y}_\ell, \kappa_j^2),$$

could be replaced by expressions of the form

$$u(\boldsymbol{x};\, f_1, \ldots, f_m) = \sum_{j=1}^{m} \int_{\partial\Omega'} e_1(\boldsymbol{x} - \boldsymbol{y}, \kappa_j^2) f_j(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}, \qquad f_1, \ldots, f_m \in C(\partial\Omega').$$

Remark 4.4. A proof of Theorem 5, in the case of approximations with respect to the C^ℓ-norms, can be found in [23].

Remark 4.5. Similar density results establishing the applicability of the MFS can be obtained for various partial differential operators including the biharmonic (and m-harmonic) and the Cauchy–Navier system in linear elasticity. See [23,25].

Acknowledgments

The author wishes to thank Professor G. Fairweather as well as the anonymous reviewers for their instructive comments. The author's work was supported by a University of Cyprus grant.

Appendix

Proof of Propostion 2. We need first the following lemmata:

Lemma A.1. *If $\mathcal{L}$ is an elliptic operator with constant coefficients of order m in $\mathbb{R}^n$ and $e = e(\boldsymbol{x})$ is a fundamental solution of $\mathcal{L}$, then $D^\alpha e \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, for every $|\alpha| < m$.*

Proof. See [32]. □

Lemma A.2. *If ν is a continuous linear functional on $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$, where $N \in \mathbb{N}$, $\sigma \in [0, 1)$, $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and the distribution $\mu = f * \nu$ has compact support, then μ defines a continuous linear functional on $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$ as well.*

Proof. Every element of $\mathscr{D}(\mathbb{R}^n)$, when restricted in $\overline{\Omega}$, defines an element of $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$, and thus the continuous linear functionals on $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$ define distributions, i.e., $\left(\mathrm{lip}^{k,\sigma}(\overline{\Omega})\right)' \subset \mathscr{D}'(\mathbb{R}^n)$. Since ν has compact support, $f * \nu$ defines also a distribution, and if $\varphi \in \mathscr{D}(\mathbb{R}^n)$, then $\nu * \check{\varphi} \in \mathscr{D}(\mathbb{R}^n)$, and

$$\langle \varphi, f * \nu \rangle = \langle (\nu * \check{\varphi})\check{}, f \rangle.$$

Note that $(\nu * \check{\varphi})\check{} \in \mathscr{D}(\mathbb{R}^n)$. Also,

$$(\nu * \check{\varphi})\check{}(\boldsymbol{x}) = (\nu * \check{\varphi})(-\boldsymbol{x}) = \langle \varphi, \tau_{\boldsymbol{x}} \nu \rangle = \langle \tau_{-\boldsymbol{x}} \varphi, \nu \rangle = \nu(\tau_{-\boldsymbol{x}} \varphi).$$

Let $\psi \in \mathscr{D}(\mathbb{R}^n)$ be such that $\psi = 1$ in a neighborhood of the support of $f * \nu$. Then

$$\langle \varphi, f * \nu \rangle = \langle \psi \varphi, f * \nu \rangle,$$

for every $\varphi \in \mathscr{D}(\mathbb{R}^n)$. Therefore,

$$\langle \varphi, f * \nu \rangle = \langle \psi \varphi, f * \nu \rangle = \int_{\mathbb{R}^n} (\nu * (\psi\varphi)\check{})\check{}(\boldsymbol{x}) f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^n} \nu\left(\tau_{-\boldsymbol{x}}(\psi\varphi)\right) f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}.$$

The last integral is well defined since $w(\boldsymbol{x}) = \nu\left(\tau_{-\boldsymbol{x}}(\psi\varphi)\right)$ is supported in

$$K = \operatorname{supp} \psi - \operatorname{supp} \nu = \{\boldsymbol{x} - \boldsymbol{y} : \boldsymbol{x} \in \operatorname{supp} \psi \text{ and } \boldsymbol{y} \in \operatorname{supp} \nu\},$$

which is compact and $w \in \mathscr{D}(\mathbb{R}^n)$. Thus,

$$|\langle \varphi, f * \nu \rangle| = \left| \int_K \nu\left(\tau_{-\boldsymbol{x}}(\psi\varphi)\right) f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \right| \le \left(\int_K |f(\boldsymbol{x})| \mathrm{d}\boldsymbol{x} \right) \cdot \sup_{\boldsymbol{x} \in K} \left|\nu\left(\tau_{-\boldsymbol{x}}(\psi\varphi)\right)\right|.$$

Also,

$$\sup_{\boldsymbol{x} \in K} \left|\nu\left(\tau_{-\boldsymbol{x}}(\psi\varphi)\right)\right| \le |\nu|_{(\mathrm{lip}^{k,\sigma}(\overline{\Omega}))'} \cdot |\psi\varphi|_{\mathrm{lip}^{k,\sigma}(\overline{\Omega})} \le c \cdot |\nu|_{(\mathrm{lip}^{k,\sigma}(\overline{\Omega}))'} \cdot |\varphi|_{\mathrm{lip}^{k,\sigma}(\overline{\Omega})},$$

where $c = \sup_{|\varphi|_{\mathrm{lip}^{k,\sigma}(\overline{\Omega})} = 1} |\psi\varphi|_{\mathrm{lip}^{k,\sigma}(\overline{\Omega})}$. Clearly, $c < \infty$. Altogether, for every $\varphi \in \mathscr{D}(\mathbb{R}^n)$, we have $|\langle \varphi, f * \nu \rangle| \le C |\varphi|_{\mathrm{lip}^{k,\sigma}(\overline{\Omega})}$, where $C = c \cdot |\nu|_{(\mathrm{lip}^{k,\sigma}(\overline{\Omega}))'} \cdot \int_K |f(\boldsymbol{x})| \mathrm{d}\boldsymbol{x}$ and thus $f * \nu$ defines an element of $\left(\mathrm{lip}^{k,\sigma}(\overline{\Omega})\right)'$. This concludes the proof of the lemma. □

Combination of Lemmata A.1 and A.2 yields the following result:

Lemma A.3. *Let $\nu \in \left(\mathrm{lip}^{k,\sigma}(\overline{\Omega})\right)'$. Let also $\mathcal{L} = \sum_{|\alpha| \le m} a_\alpha D^\alpha$ be an elliptic operator of order m with constant coefficients in $\mathbb{R}^n$ and $e = e(\boldsymbol{x})$ be a fundamental solution of $\mathcal{L}$ and $\psi \in \mathscr{D}(\mathbb{R}^n)$. If the distributions ν and $e * \nu$ have compact supports, then $\mathcal{L}\left(\psi(e * \nu)\right)$ defines an element of $\left(\mathrm{lip}^{k,\sigma}(\overline{\Omega})\right)'$.*

Proof. The distribution $\mathcal{L}\left(\psi(e * \nu)\right)$ can be expressed as

$$\begin{aligned} \mathcal{L}\left(\psi(e * \nu)\right) &= \psi \mathcal{L}(e * \nu) + \sum_{|\beta| < m} \mathcal{L}_\beta \psi (D^\beta e * \nu) \\ &= \psi \nu + \sum_{|\beta| < m} \mathcal{L}_\beta \psi (D^\beta e * \nu), \end{aligned} \tag{A.1}$$

where $\mathcal{L}_\beta$ is a partial differential operator with constant coefficients of order at most $m - |\beta|$, for every $|\beta| < m$. If $|\beta| < m$, then $D^\beta e \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, as a consequence of Lemma A.1. Also, $D^\beta e * \nu$ has compact support, for every $|\beta| < m$; in fact,

$$\operatorname{supp} D^\beta e * \nu = \operatorname{supp} D^\beta (e * \nu) \subset \operatorname{supp} e * \nu.$$

Lemma A.2 implies that $D^{\beta}e * \nu$ defines a continuous linear functional on $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$, for every $|\beta| < m$.

Note that if $\psi \in \mathscr{D}(\mathbb{R}^n)$ and $\mu \in \big(\mathrm{lip}^{k,\sigma}(\overline{\Omega})\big)'$, then $\psi\mu$ defines a continuous linear functional on $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$ by

$$(\psi\mu)(u) = \mu(\psi u).$$

Each term on the right-hand side of (A.1) defines an element of $\big(\mathrm{lip}^{k,\sigma}(\overline{\Omega})\big)'$ and so does their sum. □

We next need the following result:

Lemma A.4. *Let $\nu \in \big(\mathrm{lip}^{k,\sigma}(\overline{\Omega})\big)'$, where Ω is an open bounded subset of $\mathbb{R}^n$. Assume further that ν annihilates every u satisfying $\mathcal{L}u = 0$ in a neighborhood of $\overline{\Omega}$. If $\{U_j\}_{j=0}^{J}$ is an open covering of $\overline{\Omega}$, there exist $\{\nu^j\}_{j=0}^{J} \subset \big(\mathrm{lip}^{k,\sigma}(\overline{\Omega})\big)'$, such that* $\operatorname{supp}\nu^j \subset \overline{\Omega} \cap \overline{U}_j$, $\sum_{j=0}^{J}\nu^j = \nu$, *and each ν^j annihilates every u which satisfies $\mathcal{L}u = 0$ in a neighborhood of $\overline{\Omega} \cap \overline{U}_j$.*

Proof. Let $\{\psi^j\}_{j=0}^{J} \subset \mathscr{D}(\mathbb{R}^n)$ such that $\sum_{j=0}^{J}\psi^j = 1$ in a neighborhood of $\overline{\Omega}$ and $\operatorname{supp}\psi^j \subset U_j$, for every $j = 0, \dots, J$. If e is a fundamental solution of $\mathcal{L}$, then $\nu(\tau_{\boldsymbol{x}}e) = 0$ for every $\boldsymbol{x} \in \mathbb{R}^n \setminus \overline{\Omega}$. Also,

$$\vartheta(\boldsymbol{x}) = \nu(\tau_{\boldsymbol{x}}e) = (\check{e} * \nu)(\boldsymbol{x}).$$

Note that ϑ defines a distribution in $\mathbb{R}^n$, as a convolution of two distributions, one of which, namely ν, has compact support (i.e., $\operatorname{supp}\nu \subset \overline{\Omega}$). Further, $\mathcal{L}^*\vartheta = \nu$ in the sense of distributions, since $\check{e}$ is a fundamental solution of $\mathcal{L}^\star$ (the adjoint of $\mathcal{L}$). Also, it is clear that ϑ is a smooth function and satisfies, in the classical sense, the equation $\mathcal{L}^*u = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$. We shall show that $\operatorname{supp}\vartheta \subset \overline{\Omega}$. Let $\varphi \in \mathscr{D}(\mathbb{R}^n \setminus \overline{\Omega}) \subset \mathscr{D}$. Then

$$\langle \varphi, \check{e} * \nu \rangle = \langle (\check{e} * \varphi)\check{\ }, \nu \rangle. \tag{A.2}$$

Clearly, $(\check{e} * \varphi)\check{\ } \in C^\infty(\mathbb{R}^n)$ and $\mathcal{L}\,((\check{e} * \varphi)\check{\ }) = \check{\varphi}$, since $(\check{e} * \varphi)\check{\ } = e * \check{\varphi}$. In particular, $e * \check{\varphi}$ satisfies the equation $\mathcal{L}u = 0$, in the complement of the support of φ, thus in a neighborhood of $\overline{\Omega}$ and consequently $\langle (\check{e} * \varphi)\check{\ }, \nu \rangle = 0$. This, due to (A.2), implies that $\langle \varphi, \check{e} * \nu \rangle = 0$.

Set $\nu^j = \mathcal{L}^*(\psi^j\vartheta)$, $j = 0, \dots, J$. Lemma A.3 provides that $\{\nu^j\}_{j=0}^{J} \subset \big(\mathrm{lip}^{k,\sigma}(\overline{\Omega})\big)'$. It is clear that $\sum_{j=0}^{J}\nu^j = \nu$ and $\operatorname{supp}\nu^j \subset \overline{\Omega} \cap \overline{U}_j$. Let u be a C^∞-function defined in a neighborhood W of $\overline{\Omega} \cap \overline{U}_j$ which satisfies the equation $\mathcal{L}u = 0$. Let $\zeta \in \mathscr{D}(W)$, which is equal to one in a neighborhood of $\overline{\Omega} \cap \overline{U}_j$. Then the zero extension of ζu belongs to $\mathscr{D}(\mathbb{R}^n)$ and we have

$$\langle u, \nu^j \rangle = \langle \zeta u, \nu^j \rangle = \langle \zeta u, \mathcal{L}^*(\psi^j\vartheta) \rangle = \langle \mathcal{L}(\zeta u), \psi^j\vartheta \rangle.$$

The last term of the above is equal to zero since $\mathcal{L}(\zeta u) = 0$ in a neighborhood of the support of $\psi^j\vartheta$. Thus ν^j annihilates every u satisfying $\mathcal{L}u = 0$ in a neighborhood of $\overline{\Omega} \cap \overline{U}_j$. □

Remark A.1. Lemma A.4 remains valid if in its formulation we replace the phrase ν annihilates every u which satisfies $\mathcal{L}u = 0$ in a neighborhood of $\overline{\Omega}$ by ν annihilates $\tau_{\boldsymbol{x}}e$, for every $\boldsymbol{x} \in \mathbb{R}^n \setminus \overline{\Omega}$ where e is a fundamental solution of $\mathcal{L}$.

We are now ready to prove Proposition 2.

Since Ω satisfies the segment condition, for every $\boldsymbol{x} \in \partial\Omega$ there exist a non-zero vector $\boldsymbol{\xi}_{\boldsymbol{x}}$ and an open neighborhood $U_{\boldsymbol{x}}$ of $\boldsymbol{x}$, such that, if $\boldsymbol{y} \in U_{\boldsymbol{x}} \cap \overline{\Omega}$, then $\boldsymbol{y} + t\boldsymbol{\xi}_{\boldsymbol{x}} \in \Omega$, for every $t \in (0, 1)$. Since $\partial\Omega$ is compact, there exists a finite collection of such neighborhoods $\{U_j\}_{j=1}^J$ covering $\partial\Omega$. This collection becomes an open cover of $\overline{\Omega}$ by adding a suitable open set U_0, which can be chosen so that $\overline{U}_0 \subset \Omega$. We denote by $\boldsymbol{\xi}_j$ the non-zero vector corresponding to U_j, when $j = 1, \ldots, J$, and set $\boldsymbol{\xi}_0 = 0$. Clearly,

$$\tau_{\varepsilon\boldsymbol{\xi}_j}[U_j \cap \overline{\Omega}] \subset \Omega, \quad \text{for every } \varepsilon \in (0, 1) \text{ and } j = 0, \ldots, J.$$

Let $\{\psi^j\}_{j=0}^J \subset \mathscr{D}(\mathbb{R}^n)$ such that $\sum_{j=0}^J \psi^j = 1$ in a neighborhood of $\overline{\Omega}$ and $\operatorname{supp}\psi^j \subset U_j$, for every $j = 0, \ldots, J$. Lemma A.4, and more specifically its modified version according to Remark A.1, provides functionals $\{\nu^j\}_{j=0}^J \subset \left(\mathrm{lip}^{k,\sigma}(\overline{\Omega})\right)'$, such that $K^j = \operatorname{supp}\nu^j \subset \overline{\Omega} \cap \overline{U}_j$, $\sum_{j=0}^J \nu^j = \nu$ and each ν^j annihilates every u satisfying $\mathcal{L}u = 0$, in a neighborhood of K^j. Let $\boldsymbol{\xi}_j$ be the translation vector provided by the segment condition which corresponds to U_j. We denote by $\tau_{j,\varepsilon}$ the translation operator by $\varepsilon\boldsymbol{\xi}_j$, with $\varepsilon \in \mathbb{R}$, i.e.,

$$\left(\tau_{j,\varepsilon} \circ w\right)(\boldsymbol{x}) = \begin{cases} w(\boldsymbol{x} - \varepsilon\boldsymbol{\xi}_j) & \text{if } j = 1, \ldots, J, \\ w(\boldsymbol{x}) & \text{if } j = 0, \end{cases}$$

where w is a distribution. We also define

$$\nu_\varepsilon = \sum_{j=0}^J \nu_\varepsilon^j, \quad \text{where } \nu_\varepsilon^j = \tau_{j,\varepsilon}\nu^j.$$

Clearly, each ν_ε^j annihilates every u which satisfies $\mathcal{L}u = 0$ in a neighborhood of $\operatorname{supp}\nu_\varepsilon^j$. Also, $\operatorname{supp}\nu_\varepsilon = K_\varepsilon \subset \Omega$. Note that ν_ε annihilates every u which satisfies $\mathcal{L}u = 0$ in a neighborhood of K_ε.

Next we shall show that ν_ε converges to ν in the weak* sense of $\left(\mathrm{lip}^{k,\sigma}(\overline{\Omega})\right)'$. It suffices to show that

$$\lim_{\varepsilon\to 0} \nu_\varepsilon^j(u) = \nu^j(u), \quad \text{for every } u \in \mathrm{lip}^{k,\sigma}(\overline{\Omega}) \text{ and } j = 0, \ldots, J.$$

Let $u \in \mathrm{lip}^{k,\sigma}(\overline{\Omega})$ and $\varphi \in \mathscr{D}(\mathbb{R}^n)$. Then

$$\nu_\varepsilon^j(u) - \nu^j(u) = \left(\nu_\varepsilon^j(u) - \nu_\varepsilon^j(\varphi)\right) + \left(\nu_\varepsilon^j(\varphi) - \nu^j(\varphi)\right) + \left(\nu^j(\varphi) - \nu^j(u)\right). \tag{A.3}$$

The third term of the (A.3) can become arbitrarily small if φ is chosen sufficiently close to u and so can the first term, since $|\nu_\varepsilon^j|_{(\mathrm{lip}^{k,\sigma}(\overline{\Omega}))'} = |\nu^j|_{(\mathrm{lip}^{k,\sigma}(\overline{\Omega}))'}$. Note that the set of elements of $\mathscr{D}(\mathbb{R}^n)$, restricted in $\overline{\Omega}$, is dense in $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$. The second term of (A.3) can be written as

$$\nu_\varepsilon^j(\varphi) - \nu^j(\varphi) = \nu^j(\tau_{-\varepsilon\boldsymbol{\xi}_j}\varphi) - \nu^j(\varphi) = \nu^j(\tau_{-\varepsilon\boldsymbol{\xi}_j}\varphi - \varphi).$$

The right-hand side of the above tends to zero, as $\varepsilon \to 0$, since $\tau_{-\varepsilon\boldsymbol{\xi}_j}\varphi \to \varphi$, as $\varepsilon \to 0$, in the topology of $\mathscr{D}(\mathbb{R}^n)$, and consequently in the topology of $\mathrm{lip}^{k,\sigma}(\overline{\Omega})$ as well. The weak* convergence of ν_ε to ν, as $\varepsilon \to 0$, is now established.

Finally, let u satisfying the equation $\mathcal{L}u = 0$, in a neighborhood of $\overline{\Omega}$. Clearly, u can be viewed as an element of $\mathscr{D}'(\Omega)$, when restricted to Ω, and since $\mathcal{L}u = 0$, in Ω, then u can be viewed as a smooth function, when restricted to Ω. Therefore, $\nu_\varepsilon(u) = 0$, since u satisfies the equation $\mathcal{L}u = 0$ in a neighborhood of K_ε. Also,

$$\nu(u) = \lim_{\varepsilon \searrow 0} \nu_\varepsilon(u) = 0,$$

which concludes the proof of Proposition 2. $\square$

Remark A.2. The proof of Proposition 2, and in particular, the construction of the sequence $\{\vartheta^\ell\}_{\ell\in\mathbb{N}}$, yields the following result:

Corollary A.1. *Let $\mathcal{L}$ be an elliptic operator with constant coefficients in $\mathbb{R}^n$ of order m and Ω be an open bounded subset of $\mathbb{R}^n$. Let $k \in \mathbb{N}$, $\sigma \in [0, 1)$ and assume that the functional $\nu \in \left(\mathrm{lip}^{k,\sigma}(\overline{\Omega})\right)'$ annihilates*

$$\mathcal{Y}_{k,\sigma} = \left\{u \in C^m(\Omega) : \mathcal{L}u = 0\right\} \cap \mathrm{lip}^{k,\sigma}(\overline{\Omega}).$$

If $\{U_j\}_{j=0}^J$ is an open cover of $\overline{\Omega}$, then there exist $\{\nu^j\}_{j=0}^J \subset \left(\mathrm{lip}^{k,\sigma}(\overline{\Omega})\right)'$ annihilating $\mathcal{Y}_{k,\sigma}$ and satisfying

$$\operatorname{supp} \nu^j \subset \overline{\Omega} \cap \overline{U}_j, \quad j = 0, \ldots, J \quad \textit{and} \quad \sum_{j=0}^J \nu^j = \nu. \quad \square$$

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